International Journal of Advances in Engineering and Management (IJAEM)
Volume 3, Issue 9 Sep 2021, pp: 1391-1395 www.ijaem.net ISSN: 2395-5252

Critical Graphs W.R.T K-domination

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Submitted: 10-09-2021 Revised: 19-09-2021 Accepted: 23-09-2021

ABSTRACT: K dominating sets play an important role in Graph Theory and areas like discrete optimization. They appear in matching theory, colouring of graphs and in trees. They were introduced into the communication theory on noisy channels. The concept of K-domination was introduced by Fink and Jacobson [2] and has been studied by a number of authors. G. J. vala [3] has characterized vertices which are responsible to increase or decrease K-domination number of graphs in his thesis. It was shown by Jacobson and Peters [1] that the problem of finding a minimum K-dominating set is NP-hard. Bondage number has been discussed in [5],[6].In this chapter we have characterized edges which are responsible to increase or decrease K-domination number. We have also given some illustrations.

KEYWORDS: K dominating set, Minimal K Dominating set , Minimum K dominating set, K domination number, critical graphs, Private K neighborhood.

I. INTRODUCTION

The concept of K-domination is stronger than the concept of the domination. Actually K-domination is generalization of domination or we can say that domination is the special case of K-domination for K = 1. There are dominating sets which are not K-dominating sets for $K \geq 2$.

Definition 1.1: K-dominating Set
A set $S \subseteq V$ of vertices in a graph G = (V , E) is a K-dominating set if for every vertex $u \in V \setminus S$, $| N(u) \cap S | \geq K$

Definition 1.2 : Minimal K-dominating Set
A K-dominating Set S is said to be a minimal K-dominating set if no proper subset $S' \subseteq S$ is a K-dominating Set.

Definition 1.3 : Minimum K-dominating Set
A K-dominating set with minimum cardinality is called minimum K-dominating Set.

Definition 1.4 : K-domination number
The K domination number $\Upsilon_k(G)$ of a graph G is a cardinality of a minimum K-dominating Set in G.

Definition 1.5 : Private K-neighborhood
Let G be any graph and S be a subset of V(G). Let $v \in S$ and K ≥ 1. Then the private K-neighborhood of v with respect to S which is denoted by $PR_K\ [v,\ S]$ and defined as follows:
$PR_K[v, S] = \{w \in V(G) - S\ /\ w$ is adjacent to exactly K vertices of S including v }
∪ { v / if v is adjacent to at most K – 1 vertices of S }

Note :
(i) If S is a K-dominating set in the graph G, then $\gamma_K(G) \leq |S|$.
(ii) Every minimum K-dominating set in G is a minimal K-dominatig set in G.
(iii) If K = 1, then $\gamma_1(G) = \gamma(G)$.
(iv) Also for 1 ≤ j ≤ K, if S is a K-dominating set, then it is also j-dominating set, and therefore $\gamma_j(G) \leq \gamma_k(G)$.
(v) If G is a graph with $\Delta(G)$ (= the maximum degree of G) ≥ 3 and K ≥ 3, then $\gamma_K(G) > \gamma(G)$ [5].

Note : Let G be any graph. A K-dominating set S in the graph G is a minimal K-dominating set in G if and only if for every vertex v of the set S,
$PR_K\ [v,\ S] \neq \phi$.**[5]**

Illustration 1.6 : Here we give an example of Peterson graph which represents K = 2 dominating set.

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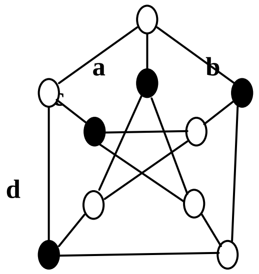

Figure 1.6.1

Here S = {a, b, c, d} is a minimum 2 dominating set. |S| = 4

So, 2 domination number of Peterson graph is 4.

Illustration 1.7 : Here we give an example of a graph which represents private K neighborhood.

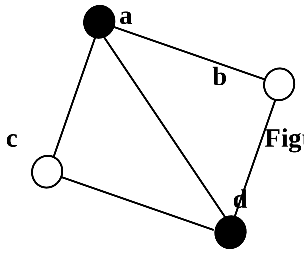

Figure 1.7.1

Here S = {a, d} is a 2-dominating set.

$PR_K[a,S]=\{a,c,b\}\neq\phi$ & $PR_K[d,S]=\{d,c,b\}\neq\phi$

Therefore S is a minimal 2-dominating set.

II EDGE REMOVAL IN K-DOMINATION

Note : It is useful to partition the Edges of G into two sets according to how their removal affects $\Upsilon_K(G)$.

We shall prove that $E = E_K^{+} \cup E_K^{O}$ for

$$E_K^{+} = \{e \in E(G) / \Upsilon_K(G-e) > \Upsilon_k(G)\}$$

$$E_K^{o} = \{e \in E(G) / \Upsilon_K(G-e) = \Upsilon_k(G)\}$$

Note : It is useful to partition the Vertices of G into two sets according to how their removal affects $\Upsilon_K(G)$.

Let $V = V_K^{+} \cup V_K^{O} \cup V_K^{-}$ **for**

$$V_K^{+} = \{v \in V(G) / \Upsilon_K(G-v) > \Upsilon_k(G)\}$$

$$V_K^{o} = \{v \in V(G) / \Upsilon_K(G-v) = \Upsilon_k(G)\}$$

$$V_K^{-} = \{v \in V(G) / \Upsilon_K(G-v) < \Upsilon_k(G)\}$$

Theorem 2.1 : An edge e = $P_1P_2 \in E_K^{+} \Leftrightarrow$ **(for every** Υ_K **Set S)** $P_1 \in S$, $P_2 \notin S$ **&** P_2 **is adjacent to exactly K vertices of S including** P_1 **.**

Proof : $\Leftarrow$ **(sufficiency)**

Let S be a Υ_K Set. Let e = P_1P_2. $P_1 \in S$, $P_2 \notin S$ & P_2 is adjacent to exactly K vertices of S including P_1. So if we remove e, then P_2 will be adjacent to exactly K-1 vertices of S. So, S will not be K-dominating Set in $G \setminus \{e\}$. This is true for every Υ_K Set S of G.

Suppose there is a K-dominating set such that $|T| \leq \gamma_K(G)$.

Case 1: Suppose $|T| = \gamma_K(G)$

First suppose that $P_1 \in T$, $P_2 \in T$. Then T is a minimum K-dominating set of graph G with $P_1 \in T$, $P_2 \in T$. This contradicts our assumption.

Suppose $P_1 \notin T$, $P_2 \notin T$. Then again T is a minimum K-dominating set of graph G not containing P_1 & P_2. Which contradicts our assumption.

Suppose $P_1 \in T$, $P_2 \notin T$. Now P_2 is adjacent to at least K vertices of T in graph G – {e}. hence P_2 is adjacent to at least K+1 vertices of T in graph G including P_1. This again contradicts our assumption.

Thus it is impossible that $|T| = \gamma_K(G)$ and T is a K-dominating set of graph G – {e}.

Case 2: Suppose $|T| < \gamma_K(G)$

Then it can be similarly proved that T is a K-dominating set of graph G with $|T| < |S|$

This is a contradiction.

There for $\Upsilon_K(G-e) > \Upsilon_k(G)$

$\Rightarrow$ **(Necessity)**

Let S be any minimum K-dominating set of graph G. Let e $\in E_K^{+}$ Means removal of edge e increases K domination number.

If P_1 & $P_2 \in S$ or P_1 & $P_2 \notin S$ then S is K dominating Set in $G-\{e\}$. Hence $\Upsilon_K(G-\{e\}) \leq \Upsilon_K(G)$ .Which is not true.

If $P_1 \in S$ & $P_2 \notin S$.

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Suppose P_2 is adjacent to at least K+1 vertices of S including P_1, then removing e, still P_2 will be adjacent to K vertices & hence e $\notin Ek^+$. Which is a contradiction.

So, e = P_1P_2 then $P_1 \in S$, $P_2 \notin S$ & P_2 is adjacent to exactly K vertices of S including P_1

▮

Remark: Above theorem can be restated as follow

Theorem 2.2 : An edge e = $P_1P_2 \in E_K^+ \Leftrightarrow$ for every minimum K-dominating set S of graph G $P_1 \in S$, $P_2 \notin S$ & $P_2 \in PR_K[P_1, S]$

Theorem 2.3 : An edge $e = P_1P_2 \in E_K^0 \Leftrightarrow$ there exist a Υ_K set S such that

(1) P_1 and $P_2 \notin S$ or P_1 and $P_2 \in S$

(2) $P_1 \in S$, $P_2 \notin S$ & P_2 is adjacent to at least K+1 vertices of S.

Proof : $\Leftarrow$ (sufficiency)

Let (1) P_1 and $P_2 \notin S$ or P_1 and $P_2 \in S$

Let T be a Υ_K Set in $G-\{e\}$. Then T is also a K dominating Set in G with |T| < |S|.

Which is a Contradiction. Therefore S is a minimum K dominating Set in $G-\{e\}$

So, $\Upsilon_K(G-\{e\}) = |S| = \Upsilon_K(G)$ means $e = P_1P_2 \in E_K^0$

Let (2) $P_1 \in S$ & $P_2 \notin S$ is adjacent to at least K+1 vertices of S. Then if we remove e then still $P_1 \in S$ and P_2 will be adjacent to K vertices of S. So, S is K dominating Set in graph G.

Let T be Υ_K Set in $G-\{e\}$ with |T| < |S|. Now, P_2 is adjacent to at least K vertices of T.

So, P_2 is adjacent to at least K+1 vertices of S. Now, if we add e = P_1P_2 then T is Υ_K set in G with |T| < |S|. Which is a contradiction. So, S is Υ_K Set in $G-\{e\}$. So, $e = P_1P_2 \in E_K^0$.

$\Rightarrow$ (Necessity)

Let e = $P_1P_2 \in E_K{}^\circ$ means removal of e does not affect K domination number.

Let S be a Υ_K Set of $G-\{e\}$. Then |S| = Υ_K (G)

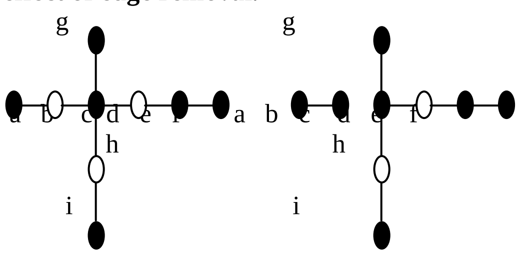
(1) P_1 and $P_2 \notin S$ or P_1 and $P_2 \in S$ then theorem is proved.

(2) Let $P_1 \in S$ and $P_2 \notin S$. Now P_2 is adjacent to at least K vertices of S in $G-\{e\}$.

Hence P_2 is adjacent to at least K+1 vertices including P_1 of S in G. Thus S is a Υ_K Set of G, Which satisfies required conditions.

▮

Illustration 2.4: Here we give an example of a graph which represents 2 dominating set and effect of edge removal.



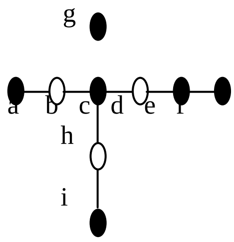


Figure 2.4.1

In above Graph G, Υ_2 set S={a, c, e, f, g, i} here |S| = 6

E_2^+ ={ab, bc, de, ch, hi}

E_2^0 ={cg, cd, ef}

III K-BONDAGE NUMBER

Definition 3.1: Bondage Number [3]

The Smallest number of edges whose removal from a graph G causes the modified graph to have a larger Domination number than G is called Bondage Number.

Definition 3. 2 : K Bondage Number

The Smallest number of edges whose removal from a graph G causes the modified graph to have a larger K-domination number than G is called K Bondage Number.

Illustration 3.3 : Here we give a table to represent 2 domination number and 2 bondage number of complete grid graph.

Graph	Domination Number	2 Domination Number	2 Bondage Number
	$\Upsilon(P_k \times P_n)$	$\Upsilon_2(P_k \times P_n)$	$\beta_2(P_k \times P_n)$
$P_2 \times P_n$	$\left\lfloor \frac{n+2}{2} \right\rfloor, n \geq 1$	$n, \ n \geq 2$	$1, n \geq 2$
$P_3 \times P_n$	$\left\lfloor \frac{3n+4}{4} \right\rfloor, n \geq 1$	$4n/3, \quad if \ n = 0 \pmod 3$ $(4n+2)/3, \ if \ n = 1 \pmod 3$ $(4n+1)/3, \ if \ n = 2 \pmod 3$ For all $n \geq 3$	$2, n = 3k+1$ $k \geq 1$ $1 \quad otherwise$
$P_4 \times P_n$	n+1 n=1,2,3,5,6,9 n otherwise for $n \geq 1$	2n for $n \geq 2$	2 $n \neq 1,2$
$P_5 \times P_n$	$\left\lfloor \frac{6n+6}{5} \right\rfloor$ $n = 2, 3, 7$ $\left\lfloor \frac{6n+8}{5} \right\rfloor$ $otherwise \ n \geq 1$	$\frac{5n}{2} \quad ; n = 0 \pmod 2$ $\frac{5n-1}{2} \quad ; n = 1 \pmod 2$ $n \geq 2$	1 n = even 2 n = odd

IV CONCLUSION AND FURTHER SCOPE OF RESEARCH

In this paper, we have derived characterisation of edges whose removal affects or does not affects K-domination number. Illustrations have been given to understand theorems clearly. This will be helpful for the readers who are working on various domination related parameters w.r.t critical graphs. One may try for edge addition also. K bondage number also have been discussed which will be useful to define relation between vertices and edges. According to the application, one may proceed for the further research regarding corresponding variants of domination.

It will be interesting if one can find relation between vertices and edges whose removal affects K domination. Moreover bondage number and reinforcement number can enrich content for classes of graphs.

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